ROBUST STABILIZATION OF UNCERTAIN HIGH ORDER SYSTEMS VIA SMOOTH OUTPUT FEEDBACK

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Abstract: We investigate the robust stabilization problem for a family of uncertain nonlinear systems with uncontrollable/unobservable linearization. To achieve global robust stabilization using a single output feedback controller, we introduce a rescaling transformation with an appropriate dilation, which turns out to be very effective in dealing with uncertainty of the system. Using this rescaling technique combined with the non-separation principle based design methods (Qian and Lin, 2002*a*; Yang and Lin, 2003), we develop a robust output feedback control scheme for uncertain nonlinear systems satisfying a homogeneous growth condition. Both a smooth state feedback controller and a homogeneous observer are designed for the rescaled system using only the knowledge of the bounding system rather than the uncertain system itself.

Keywords: Global robust stabilization, Homogeneous observers, Non-separation principle design, Rescaling transformation, Smooth output feedback, Uncontrollable/unobservable linearization, Uncertain nonlinear systems.

1. INTRODUCTION

In this paper, we consider the problem of global simultaneous stabilization via a single output feedback controller, for a family of uncertain high-order systems of the form

$$\begin{aligned}\dot{\eta}_1 &= \eta_2^p + \phi_1(t,\eta,v)\\ &\ \vdots\\ \dot{\eta}_{n-1} &= \eta_n^p + \phi_{n-1}(t,\eta,v)\\ \dot{\eta}_n &= v + \phi_n(t,\eta,v)\\ y &= \eta_1, \end{aligned}\qquad (1.1)$$

where $v \in \mathbb{R}$, $\eta \in \mathbb{R}^n$ and $y \in \mathbb{R}$ are the system input, state and output, respectively, and $p \geq 1$ is an *odd* integer. The mappings $\phi_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, n$, represent a class of C^1 functions that involve uncertainty and may not be precisely known.

It is worth to mention that a necessary and sufficient condition has been characterized recently (Cheng and Lin, 2003) for the existence of a change of coordinates (diffeomorphism) and a state feedback control law transforming a smooth affine system

$$\dot{\xi} = f(\xi) + g(\xi)w \qquad \text{and} \qquad y = h(\xi),$$

into the nonlinear system (1.1) with a suitable form of $\phi_i(t,\eta,v)$ (see (Cheng and Lin, 2003) for details). Thus, (1.1) can be viewed as a generalized normal form of affine systems when exact feedback linearization is not possible. It is referred as a p—normal form (Cheng and Lin, 2003)

For the class of affine systems that is topologically equivalent to the p—normal form (1.1), interesting stabilization results have been obtained over the years. For example, local and global asymptotic stabilization of the system (1.1) with $n = 3$, $p = 3$ and $\phi_i(t,\eta,v) = 0$, $i = 1, 2, 3$, by means of smooth state feedback, were investigated in (Crouch and Irving, 1983) and (Byrnes and Isidori, 1989), respectively. In the n-dimensional case, a globally stabilizing smooth state feedback control law was explicitly designed by the tool of adding a power integrator (Lin and Qian, 2000), for a class of nonlinear systems (1.1) under appropriate growth conditions that can be regarded as a high-order version of feedback linearizable condition.

Much of the literature on stabilization of nonlinear systems has focused on the design of state feedback. In the past two years, research efforts towards the development of output feedback control schemes for the nonlinear system (1.1) have gained momentum. The paper (Qian and Lin, 2002*b*) studied the global stabilization of the high-order system (1.1) in the plane using smooth output feedback. Under suitable conditions imposed on $\phi_i(\cdot)$, a reduced-order nonlinear observer was designed in (Qian and Lin, 2002*b*), resulting in a globally stabilizing, smooth dynamic output compensator. Notably, the output feedback design in (Qian and Lin, 2002*b*) does not rely on the separation principle. Instead, it uses the idea of coupled controller-

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observer construction. In (Dayawansa, 2002), Dayawansa proved the existence of a smooth output feedback stabilizer for the three-dimensional system (1.1) when $p = 3$ and $\phi_i(\cdot) \equiv 0, i = 1, 2, 3$. His proof is based on the theory of homogeneous systems (Hahn, 1967; Bacciotti, 1992) and some elegant design techniques from (Dayawansa, 1992; Dayawansa *et al.*, 1990; Kawski, 1989; Hermes, 1991; Rosier, 1992).

More recently, we have shown that for the n-dimensional nonlinear system (1.1) with $\phi_i(\cdot) = 0$, $i = 1, \cdots, n$, which is homogeneous, the problem of global stabilization is solvable by smooth output feedback (Yang and Lin, 2003). This was done by developing a new observe design technique for the construction of a homogeneous observer, combined with the tool of adding a power integrator (Lin and Qian, 2000) for the design of a smooth state feedback controller. When $\phi_i(\cdot) \neq 0$, $i = 1, \cdots, n$ and satisfy a global Lipschitz-like condition, we further showed that global stabilization of the non-homogeneous system (1.1) can still be achieved via smooth output feedback (Yang and Lin, 2003). A key ingredient of the output feedback control strategy in (Yang and Lin, 2003) is the development of a recursive algorithm for the design of homogeneous observers, which makes it possible to assign the gains of the homogeneous observer one-by-one, in a step-by-step manner. Although such an observer design is substantially different from the "Luenberger" or "high-gain" observer design (Khalil and Saberi, 1987; Gauthier *et al.*, 1992; Krener and Isidori, 1983; Isidori, 1999; Krener and Kang, 2003), it still uses a copy of the original system and hence requires the precise information of the controlled plant. In other words, the nonlinear functions $\phi_i(t, \eta, v)$, $i = 1, \cdots, n$, in (1.1) must be independent of t and involve no uncertainty. As a result, the output feedback control scheme in (Yang and Lin, 2003) is not robust with respect to parametric or structural uncertainty and can only be applied to a single nonlinear system with an accurate model.

The purpose of this paper is two-folds: to address the robust issue discussed above, and to develop a robust output feedback control scheme for a family of uncertain nonlinear systems (1.1) that satisfies the following homogeneous growth condition:

Assumption 1.1. There exists a real constant $C \geq 0$ such that $\forall (t, \eta, v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$,

$$|\phi_i(t, \eta, v)| \leq C\Big(|\eta_1|^p + \cdots + |\eta_i|^p\Big), \quad i = 1, \cdots, n. \quad (1.2)$$

The control objective is to find, if possible, a single smooth dynamic output compensator

$$\dot{\hat{x}} = \theta(\hat{x}, y), \qquad \hat{x} \in \mathbb{R}^{n-1},$$
$$v = v(\hat{x}, y), \quad (1.3)$$

that globally simultaneously stabilizes the entire family of uncertain systems (1.1). Since Assumption 1.1 is weaker than the high-order type of the global Lipschitz-like condition given in (Yang and Lin, 2003), the class of nonlinear systems considered in this paper is larger than the one studied in (Yang and Lin, 2003). More significantly, because the feedback design of the dynamic output compensator (1.3) uses only the knowledge of the upper-bound of $\phi_i(\cdot)$ (i.e., the condition (1.2)) instead of $\phi_i(\cdot)$ itself, global output feedback stabilization will be achieved in a robust fashion, that is, in a manner which is not sensitive to perturbations and parametric uncertainties in the system. The key for such robustness is the introduction of a rescaling technique with a subtle dilation, which transforms the original system (1.1) into a rescaled one for which a dynamic output compensator can be constructed using the output feedback design method in (Yang and Lin, 2003), with a suitable twist. With the help of the rescaling technique, the uncertain nonlinearities $\phi_i(\cdot)$ in (1.1) can be handled easily by tuning the rescaling factor. In the case of uncertain systems with controllable/observable linearization (i.e., $p = 1$), the new design method provides not only a deeper insight but also an interesting alternative solution to the output feedback stabilization problem considered in (Qian and Lin, 2002*a*).

2. THE CASE WHEN $P = 1$

To better understand how the problem global robust stabilization of the uncertain nonlinear system (1.1) can be solved by output feedback under Assumption 1.1, we revisit a simple situation of (1.1) where $p = 1$, i.e., the case when the first approximation of (1.1) is controllable and observable. In this case, the uncertain system (1.1) can be rewritten as

$$\begin{aligned}
\dot{\eta}_1 &= \eta_2 + \phi_1(t, \eta, v) \\
&\vdots \\
\dot{\eta}_{n-1} &= \eta_n + \phi_{n-1}(t, \eta, v) \\
\dot{\eta}_n &= v + \phi_n(t, \eta, v) \\
y &= \eta_1, \qquad (2.1)
\end{aligned}$$

and Assumption 1.1 reduces to the linear growth condition:

$$|\phi_i(t, \eta, v)| \leq C\Big(|\eta_1| + \cdots + |\eta_i|\Big), \quad i = 1, \cdots, n. \quad (2.2)$$

In the paper (Qian and Lin, 2002*a*), we have shown that global robust stabilization of the uncertain system (2.1) satisfying (2.2) is solvable by a linear output dynamic compensator. The proof was not based on the separation principle but instead, it relied on a coupled controller-observer design (Qian and Lin, 2002*a*). Due to the intricacy of such a design, it is, however, not easy to extend the output feedback design approach of (Qian and Lin, 2002*a*) to a family of high-order uncertain systems such as (1.1).

In this section, we explore an alternative output feedback control strategy that takes advantage of homogeneity of the system, and hence can be naturally extended, in an intuitive and transparent manner, to the high-order uncertain system (1.1) with $p > 1$. To this end, we introduce a rescaling transformation with a suitable dilation for the original system (2.1), which turns out to be crucial for dominating the of uncertainty (2.1). To be precise, let

$$x_i = \frac{\eta_i}{M^{i-1}}, \quad i = 1, \cdots, n, \quad \text{and} \quad u = \frac{v}{M^n}, \quad (2.3)$$

where $M > 1$ is a rescaling factor to be determined later.

Under the new coordinates x_i's, the uncertain system (2.1) can be expressed as

$$\begin{aligned}
\dot{x}_1 &= Mx_2 + f_1(t, x, u) \\
&\vdots \\
\dot{x}_{n-1} &= Mx_n + f_{n-1}(t, x, u) \\
\dot{x}_n &= Mu + f_n(t, x, u) \\
y &= x_1, \qquad (2.4)
\end{aligned}$$

By the hypothesis (2.2) and the fact $M > 1$,

$$\begin{aligned}
|f_i(\cdot)| = \left|\frac{\phi_i(\cdot)}{M^{i-1}}\right| &\leq \frac{C(|\eta_1| + \cdots + |\eta_i|)}{M^{i-1}} \\
&\leq C\left(|x_1| + \cdots + |x_i|\right), \ 1 \leq i \leq n. \ (2.5)
\end{aligned}$$

For the rescaled uncertain system (2.4) with the constraint (2.5), it is easy to design a linear state feedback controller

$$x^*_{n+1} = -\beta_n \xi_n = -(b_1 x_1 + \cdots + b_n x_n), \quad (2.6)$$

such that

$$\dot{U}_n \le -M\Big[3(\xi_1^2+\cdots+\xi_n^2)-\xi_n(u-x_{n+1}^*)\Big], \qquad (2.7)$$

where $U_n = \frac{1}{2}(\xi_1^2+\cdots+\xi_n^2)$ is a quadratic Lyapunov function, $\xi_i = x_i - x_i^*,\ i=1,\cdots,n$, and

$$x_1^*=0, \quad x_2^* = -\beta_1\xi_1, \quad \cdots, \quad x_n^* = -\beta_{n-1}\xi_{n-1},$$

with β_i and b_i being known constants independent of M.

Next, we apply the idea of designing a reduced-order observer (Yang and Lin, 2003) to get the linear observer

$$\begin{aligned}\dot{\hat{z}}_2 &= M(\hat{z}_3+L_3L_2x_1)-ML_2(\hat{z}_2+L_2x_1)\\ &\ \vdots\\ \dot{\hat{z}}_n &= Mu - ML_n\cdots L_2(\hat{z}_2+L_2x_1),\end{aligned} \qquad (2.8)$$

which, in turn, leads to the following estimate for the unmeasurable state $(x_2,\cdots,x_n)$:

$$\hat{x}_i = \hat{z}_i + L_i\cdots L_2x_1, \qquad i=2,\cdots,n. \qquad (2.9)$$

The gain parameters $L_2,\cdots,L_n$ are to be assigned later.

It should be pointed out that the reduced order observer (2.8) is different from the one in (Yang and Lin, 2003) in two respects: 1) it does not involve a copy of $\phi_i(\cdot)$ which is an uncertain function and hence not implementable. In other words, the new observer (2.8) needs no precise information of the uncertainty of the system; 2) it contains a rescaling factor M that turns out to be very useful in dominating the uncertainty of the system, under the linear growth condition (2.2).

Let $e_i = x_i - \hat{x}_i = (x_i - L_i\cdots L_2x_1)-\hat{z}_i,\ 2\le i\le n$, be the estimate errors. Then, the error dynamics is give by

$$\begin{aligned}\dot{e}_2 &= Me_3 + f_2(\cdot) - ML_2e_2 - L_2f_1(\cdot)\\ &\ \vdots\\ \dot{e}_n &= f_n(\cdot) - ML_n\cdots L_2e_2 - L_n\cdots L_2f_1(\cdot).\end{aligned} \qquad (2.10)$$

Inspired by the certainty equivalence principle, we replace the unmeasurable state $(x_2,\cdots,x_n)$ in the controller (2.6) by its estimate $(\hat{x}_2,\cdots,\hat{x}_n)$ generated by the observer (2.8)-(2.9). In this way, we get the implementable controller

$$u = -(b_1x_1+b_2\hat{x}_2+\cdots+b_n\hat{x}_n). \qquad (2.11)$$

Substituting (2.11) into (2.7) yields

$$\dot{U}_n \le -M\Big[2(\xi_1^2+\cdots+\xi_n^2)-K(e_2^2+\cdots+e_n^2)\Big]. \qquad (2.12)$$

where $K>0$ is a fixed generic constant independent of M.

Now, consider the Lyapunov function

$$V_n = U_n + \frac{1}{2}\Big[e_2^2+(e_3-L_3e_2)^2+\cdots+(e_n-L_ne_{n-1})^2\Big].$$

Using the completion of square, together with the linear growth condition (2.2), it is not difficult to prove that

$$\begin{aligned}\dot{V}_n \le -M\Big(&\Sigma_{i=1}^n\xi_i^2 + \Big[L_2 - c_2(L_3,\cdots,L_n) - \frac{K_2L_2^2}{M}\Big]e_2^2\\ &+\cdots+\Big[L_{n-1}-c_{n-1}(L_n)-\frac{K_{n-1}L_{n-1}^2}{M}\Big]e_{n-1}^2\\ &+\Big[L_n - c_n - \frac{K_nL_n^2}{M}\Big]e_n^2\Big),\end{aligned} \qquad (2.13)$$

where $c_2(L_3,\cdots,L_n),\cdots,c_{n-1}(L_{n-1},L_n),c_{n-1}(L_n)$ are positive constants independent of M, $c_n>0$ and $K_i>0, 2\le i\le n$ are known constants independent of M and all L_i's.

Choosing the gain parameters L_i and M one-by-one, in the order of $L_n,\cdots,L_2,M$ as follows:

$$\begin{aligned}L_n - c_n - K_n \ge 1 &\Rightarrow L_n \ge 1+c_n+K_n\\ L_{n-1}-c_{n-1}(L_n)-K_{n-1}\ge 1 &\Rightarrow L_{n-1}\ge 1+c_{n-1}(L_n)+K_{n-1}\\ &\ \vdots\\ L_2 - c_2(L_3,\cdots,L_n)-K_2\ge 1 &\Rightarrow L_2\ge 1+c_2(L_3,\cdots,L_n+K_2\\ M\ge \max\{L_2^2,&\cdots,L_n^2\},\end{aligned} \qquad (2.14)$$

we have

$$\dot{V}_n(\cdot) \le -M\Big[(\xi_1^2+\cdots+\xi_n^2)+(e_2^2+\cdots+e_n^2)\Big].$$

Hence, the nonlinear system (2.1) is globally asymptotically stabilizable via output feedback.

3. THE CASE WHEN $P \ge 1$

The robust output feedback control scheme developed so far for the uncertain system (2.1) with controllable/observable linearization can be extended, in a natural manner, to its high-order counterpart. In this section, we show that in spite of loss of controllability and observability in the first approximation, a robust output feedback control method can be developed for a family of uncertain high-order systems (1.1) under Assumption 1.1. In particular, with the help of Lemmas 6.1—6.5 in the appendix, it is possible to establish the following output feedback stabilization theorem that is the main result of this paper.

Theorem 3.1. For a family of uncertain systems (1.1) satisfying Assumption 1.1, there exists a smooth output feedback controller of the form (1.3), such that the closed-loop system (1.1)–(1.3) is globally asymptotically stable at the equilibrium $(x,\hat{x})=(0,0)$.

Proof: We shall prove this theorem by explicitly designing a robust smooth state feedback controller, and a homogeneous observer that does not require the information of the uncertainties (i.e., $\phi_i(t,\eta,v),\ i=1,\cdots,n$). The construction of the robust observer is significantly differently from the one (Yang and Lin, 2003) in the sense that no copy of $\phi_i(t,\eta,v)$ appears in the robust observer, while the nonlinear observer in (Yang and Lin, 2003) did use a copy of $\phi_i(t,\eta,v)$, and thus $\phi_i(t,\eta,v)$ in (Yang and Lin, 2003) must be known precisely and time-independent. Another new ingredient of our output feedback design is the development a novel technique for handling the uncertain terms in the system (1.1). In particular, a higher-order rescaling transformation with a subtly selected dilation is employed to govern the uncertainty.

For the convenience of the reader, we break up the proof into three parts.

Part 1 — Rescaling of the Original System. When $\phi_i(\cdot)=0$, system (1.1) becomes a homogeneous system with dilation $(1,\cdots,1;p)$ and degree $p-1$ (see, for example, (Kawski, 1989; Hermes, 1991)). Keeping this in mind and motivated by the rescaling technique used in the last section, we introduce the following rescaling transformation

$$\begin{aligned}x_1 &= \eta_1\\ x_2 &= \frac{\eta_2}{M^{\frac{1}{p}}}\\ &\ \vdots\\ x_n &= \frac{\eta_n}{M^{\frac{1}{p}+\cdots+\frac{1}{p^{n-1}}}}\\ u &= \frac{v}{M^{1+\frac{1}{p}+\cdots+\frac{1}{p^{n-1}}}},\end{aligned} \qquad (3.1)$$

with dilation $(0,\frac{1}{p},\cdots,\frac{1}{p}+\cdots+\frac{1}{p^{n-1}};1+\frac{1}{p}+\cdots+\frac{1}{p^{n-1}})$, where $M\ge 1$ is a rescaling factor to be assigned later.

In the rescaled coordinates x_i's, the uncertain system can be represented as

$$\begin{aligned}\dot{x}_1 &= Mx_2^p + f_1(\cdot)\\ &\vdots\\ \dot{x}_{n-1} &= Mx_n^p + f_{n-1}(\cdot)\\ \dot{x}_n &= Mu + f_n(\cdot),\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}|f_1(\cdot)| &= |\phi_1(\cdot)| \le C|\eta_1|^p \le C|x_1|^p\\ |f_2(\cdot)| &= \left|\frac{\phi_2(\cdot)}{M^{\frac{1}{p}}}\right| \le \frac{C(|\eta_1|^p + |\eta_2|^p)}{M^{\frac{1}{p}}}\\ &\le CM^{1-\frac{1}{p}}\left(|x_1|^p + |x_2|^p\right)\\ &\vdots\\ |f_n(\cdot)| &= \left|\frac{\phi_n(\cdot)}{M^{\frac{1}{p}+\cdots+\frac{1}{p^{n-1}}}}\right| \le \frac{C(|\eta_1|^p + \cdots + |\eta_n|^p)}{M^{\frac{1}{p}+\cdots+\frac{1}{p^{n-1}}}}\\ &\le CM^{1-\frac{1}{p^{n-1}}}\left(|x_1|^p + \cdots + |x_n|^p\right).\end{aligned} \tag{3.3}$$

The estimates obtained in (3.3) are a direct consequence of Assumption 1.1 and the fact $M \ge 1$.

In this way, a new design parameter — the rescaling factor M — is added into the design of a dynamic output compensator for the rescaled system (3.2). It creates an extra freedom and will play an important role in dominating the uncertainty of the system, i.e., $f_i(\cdot),\ 1 \le i \le n$, in (3.2).

Part 2 — State Feedback Design. For the rescaled system (3.2) satisfying the growth condition (3.3), we now construct a robust state feedback controller via the adding a power integrator method (Lin and Qian, 2000). Indeed, following the design procedure of (Lin and Qian, 2000), it is easy to obtain

$$\begin{aligned}&x_1^* = 0, && \xi_1 = x_1 - x_1^*, && U_1 = \frac{\xi_1^2}{2},\\ &x_2^* = -\beta_1\xi_1, && \xi_2 = x_2 - x_2^*, && U_2 = U_1 + \frac{\xi_2^2}{2},\\ &\vdots && \vdots && \vdots\\ &x_n^* = -\beta_{n-1}\xi_{n-1}, && \xi_n = x_n - x_n^*, && U_n = U_{n-1} + \frac{\xi_n^2}{2},\end{aligned}$$

and a smooth state feedback control law

$$x_{n+1}^* = -(\beta_n\xi_n)^p = -(b_1x_1 + \cdots + b_nx_n)^p \tag{3.4}$$

such that

$$\dot{U}_n(\xi_1, \cdots, \xi_n) \le M\Big[-6(\sum_{i=1}^{n}\xi_i^{p+1}) + \xi_n(u - x_{n+1}^*)\Big] \tag{3.5}$$

where all the constants $\beta_1, \cdots, \beta_n$ and $b_1, \cdots, b_n$ are known and independent of M.

Part 3 — Output Feedback Design. Since $y = x_1$ is measurable and only unmeasurable states of the rescaled system (3.2) are $(x_2, \cdots, x_n)$, we need only to design a reduced-order $(n-1)$-dimensional observer for (3.2). However, the observer design method in (Yang and Lin, 2003) cannot be applied to the rescaled system (3.2) as it uses a copy of $f_i(\cdot), i = 1, \cdots, n$, which are time-varying and not precisely known. Motivated by the robust observer design in Section 2, next we shall design an $(n-1)$-dimensional robust homogeneous observer to estimate, instead of the states $(x_2, \cdots, x_n)$, the unmeasurable variables

$$\begin{aligned}z_2 &= x_2 - L_2x_1\\ &\vdots\\ z_n &= x_n - L_n\cdots L_2x_1,\end{aligned} \tag{3.6}$$

where the parameters $L_2, \cdots, L_n > 0$ are gain constants to be determined later.

From (3.6) it follows that

$$\begin{aligned}\dot{z}_2 &= (Mx_3^p + f_2(\cdot)) - L_2(Mx_2^p + f_1(\cdot))\\ &\vdots\\ \dot{z}_n &= (Mu + f_n(\cdot)) - L_n\cdots L_2(Mx_2^p + f_1(\cdot)).\end{aligned} \tag{3.7}$$

In view of (3.7), one can construct (similar to the previous section) the $(n-1)$–dimensional nonlinear observer

$$\begin{aligned}\dot{\hat{z}}_2 &= M(\hat{z}_3 + L_3L_2x_1)^p - ML_2(\hat{z}_2 + L_2x_1)^p\\ &\vdots\\ \dot{\hat{z}}_n &= Mu - ML_n\cdots L_2(\hat{z}_2 + L_2x_1)^p,\end{aligned} \tag{3.8}$$

which does not involve the uncertain functions $f_1(\cdot), \cdots, f_n(\cdot)$ in (3.2). This is substantially different from the homogeneous observer proposed in (Yang and Lin, 2003).

By construction, the reduced-order observer (3.8) is implementable. Moreover, the estimates of x_i's can be obtained based on the following relationships:

$$\hat{x}_i = \hat{z}_i + L_i\cdots L_2x_1, \qquad i = 2, \cdots, n. \tag{3.9}$$

Let $e_i = z_i - \hat{z}_i = x_i - \hat{x}_i,\ i = 2, \cdots, n$, be the estimate errors. Then, the error dynamics can be written as

$$\begin{aligned}\dot{e}_2 &= M(x_3^p - \hat{x}_3^p) + f_2(\cdot) - ML_2(x_2^p - \hat{x}_2^p) - L_2f_1(\cdot)\\ &\vdots\\ \dot{e}_n &= f_n(\cdot) - ML_n\cdots L_2(x_2^p - \hat{x}_2^p) - L_n\cdots L_2f_1(\cdot)\end{aligned} \tag{3.10}$$

By the certainty equivalence principle, the unmeasurable state $(x_2, \cdots, x_n)$ in the controller (3.4) is replaced by its estimate $(\hat{x}_2, \cdots, \hat{x}_n)$ generated by the nonlinear observer (3.8)–(3.9). In this way, one obtains the implementable feedback controller

$$u = -\left(b_1x_1 + b_2\hat{x}_2 + \cdots + b_n\hat{x}_n\right)^p, \tag{3.11}$$

where $\hat{x}_i = \hat{z}_i + L_i\cdots L_2x_1, \quad 2 \le i \le n$.

Substituting (3.11) into (3.5) and using Lemmas 6.3 and 5.1, we have (with a tedious but direct calculation)

$$\dot{U}_n \le -M\Big[5(\sum_{i=1}^{n}\xi_i^{p+1}) - K(\sum_{i=2}^{n}e_i^{p+1})\Big], \tag{3.12}$$

where $K > 0$ is a fixed generic constant independent of M.

To determine the observer gains $L_2, \cdots, L_n$, we introduce the change of coordinates

$$\tilde{e}_2 = e_2,\ \tilde{e}_3 = e_3 - L_3e_2,\ \cdots,\ \tilde{e}_n = e_n - L_ne_{n-1} \tag{3.13}$$

In the coordinates of ξ and $\tilde{e}$, (3.12) can be represented as (by Lemma 6.3)

$$\begin{aligned}\dot{U}_n \le -M\Big[&5(\Sigma_{i=1}^{n}\xi_i^{p+1}) - c_2(L_3, \cdots, L_n)\tilde{e}_2^{p+1}\\ &- \cdots - c_{n-1}(L_n)\tilde{e}_{n-1}^{p+1} - c_n\tilde{e}_n^{p+1}\Big],\end{aligned} \tag{3.14}$$

where $c_2(L_3, \cdots, L_n), \cdots, c_{n-2}(L_{n-1}, L_n), c_{n-1}(L_n)$ are positive real constants independent of M, and $c_n > 0$ is a known constant independent of M and all the L_i's.

On the other hand, the error dynamics (3.10) can be expressed as (in the coordinate $\tilde{e}$)

$$\begin{aligned}\dot{\tilde{e}}_2 &= M(x_3^p - \hat{x}_3^p) + f_2(\cdot) - ML_2(x_2^p - \hat{x}_2^p) - L_2f_1(\cdot)\\ &\vdots\\ \dot{\tilde{e}}_{n-1} &= M(x_n^p - \hat{x}_n^p) + f_{n-1}(\cdot) - ML_{n-1}(x_{n-1}^p - \hat{x}_{n-1}^p)\\ &\quad -L_{n-1}f_{n-2}(\cdot)\\ \dot{\tilde{e}}_n &= f_n(\cdot) - ML_n(x_n^p - \hat{x}_n^p) - L_nf_{n-1}(\cdot).\end{aligned} \tag{3.15}$$

For system (3.15), consider the Lyapunov function

$$W_n(\tilde{e}_2,\cdots,\tilde{e}_n)=\frac{1}{2}(\tilde{e}_2^2+\cdots+\tilde{e}_n^2).$$

Then, a straightforward computation gives

$$\dot{W}_n=-M\Big(\Sigma_{i=2}^n L_i\tilde{e}_i\left[(\hat{x}_i+\tilde{e}_i)^p-\hat{x}_i^p\right]+\Sigma_{i=3}^n\tilde{e}_{i-1}(x_i^p-\hat{x}_i^p)$$
$$-\Sigma_{i=2}^n L_i\tilde{e}_i\left[x_i^p-(\hat{x}_i+\tilde{e}_i)^p\right]+\Sigma_{i=2}^n\tilde{e}_i\frac{f_i(\cdot)}{M}$$
$$-\Sigma_{i=2}^n L_i\tilde{e}_i\frac{f_{i-1}(\cdot)}{M}\Big). \tag{3.16}$$

Similar to (Yang and Lin, 2003), it is not difficult to get the following estimations for each term on the right hand side of (3.16):

$$\left|\tilde{e}_{i-1}(x_i^p-\hat{x}_i^p)\right|\le\frac{\xi_i^{p+1}+\xi_{i-1}^{p+1}}{n-1}+\hat{c}_2(\cdot)\tilde{e}_2^{p+1}+\cdots+\hat{c}_i(\cdot)\tilde{e}_i^{p+1},$$
$$\left|L_i\tilde{e}_i[x_i^p-(\hat{x}_i+\tilde{e}_i)^p]\right|\le\frac{\xi_i^{p+1}+\xi_{i-1}^{p+1}}{n-1}+\hat{c}_2(\cdot)\tilde{e}_1^{p+1}+\cdots+\hat{c}_{i-1}(\cdot)\tilde{e}_i^{p+1},$$
$$\left|\tilde{e}_i\frac{f_i(\cdot)}{M}\right|\le\frac{1}{n-1}(\Sigma_{j=1}^i\xi_j^{p+1})+K\tilde{e}_i^{p+1}$$
$$\left|L_i\tilde{e}_i\frac{f_{i-1}(\cdot)}{M}\right|\le\frac{1}{n-1}(\Sigma_{j=1}^{i-1}\xi_j^{p+1})+KM^{-\frac{1}{p^{i-2}}}L_i^{p+1}\tilde{e}_i^{p+1}, \tag{3.17}$$

where $i=2,\cdots,n$ and $\hat{c}_2(\cdot)\triangleq\hat{c}_2(L_3,\cdots,L_n),\cdots,\hat{c}_{n-1}(\cdot)\triangleq\hat{c}_{n-1}(L_n)$ are positive constants independent of M. Moreover, both $\hat{c}_n>0$ and $K>0$ are known constants independent of L_i's and M.

Substituting (3.17) into (3.16), one can deduce from Lemma 6.4 that

$$\dot{W}_n\le M\Big((\Sigma_{i=1}^n\xi_i^{p+1})-\frac{1}{2^{p-1}}(\Sigma_{i=2}^n L_i\tilde{e}_i^{p+1})$$
$$+\left[\tilde{c}_2(L_3,\cdots,L_n)+K_2M^{-1}L_2^{p+1}\right]\tilde{e}_2^{p+1}$$
$$+\cdots+\left[\tilde{c}_{n-1}(L_n)+K_{n-1}M^{-\frac{1}{p^{n-3}}}L_{n-1}^{p+1}\right]\tilde{e}_{n-1}^{p+1}$$
$$+\left[\tilde{c}_n+K_nM^{-\frac{1}{p^{n-2}}}L_n^{p+1}\right]\tilde{e}_n^{p+1}\Big) \tag{3.18}$$

where $\tilde{c}_2(L_3,\cdots,L_n),\cdots,\tilde{c}_{n-1}(L_{n-1},L_n),\tilde{c}_{n-1}(L_n)$ are positive constants independent of M, while $\tilde{c}_n>0$ and $K_i>0, 2\le i\le n$ are known constants independent of L_i's and M.

Finally, choose the Lyapunov function

$$V_n(\xi_1,\cdots,\xi_n,\tilde{e}_2,\cdots,\tilde{e}_n)=U_n(\xi_1,\cdots,\xi_n)+W_n(\tilde{e}_2,\cdots,\tilde{e}_n)$$

for the closed-loop system in the coordinates $(\xi,\tilde{e})$. Then, it is straightforward to show that

$$\dot{V}_n\le-M\Big(\left[\xi_1^{p+1}+\cdots+\xi_n^{p+1}\right]$$
$$+\Big[\frac{L_2}{2^{p-1}}-C_2(L_3,\cdots,L_n)-\frac{K_2}{M}L_2^{p+1}\Big]\tilde{e}_2^{p+1}$$
$$+\cdots+\Big[\frac{L_{n-1}}{2^{p-1}}-C_{n-1}(L_n)-\frac{K_{n-1}}{M^{\frac{1}{p^{n-3}}}}L_{n-1}^{p+1}\Big]\tilde{e}_{n-1}^{p+1}$$
$$+\Big[\frac{L_n}{2^{p-1}}-C_n-\frac{K_n}{M^{\frac{1}{p^{n-2}}}}L_n^{p+1}\Big]\tilde{e}_n^{p+1}\Big), \tag{3.19}$$

where $C_2(L_3,\cdots,L_n),\cdots,\mathrm{C}_{n-2}(L_{n-1},L_n),C_{n-1}(L_n)$ are positive constants independent of M, while $C_n>0$ and $K_i>0,\ 2\le i\le n$ are positive constants independent of L_i's and M.

From (3.19), it is easy to conclude that if the gain parameters L_i's and M are assigned one-by-one, in the order of $L_n,L_{n-1},\cdots,L_2,M$ (i.e., in a manner similar to (2.14)), one has

$$\dot{V}_n\le-M\left[(\xi_1^{p+1}+\cdots+\xi_n^{p+1})+(\tilde{e}_2^{p+1}+\cdots+\tilde{e}_n^{p+1})\right].$$

This, in turn, implies that the uncertain nonlinear system (1.1) is globally asymptotically stabilized by the dynamic output compensator (3.8)–(3.11). □

So far, we have considered the problem of global output feedback stabilization for the uncertain high order system (1.1) satisfying a lower-triangular growth condition. In the remainder of this section, we briefly discuss to what extent, the output feedback stabilization result thus obtained can be extended to a larger class of uncertain nonlinear systems in the p-normal form (Cheng and Lin, 2003), which goes beyond a *strict-triangular* structure.

As shown in (Cheng and Lin, 2003), every smooth affine system is, under appropriate conditions, feedback equivalent to the following nonlinear system

$$\dot{\eta}_1=\eta_2^p+\eta_2^{p-1}\phi_{1,p-1}(t,\eta,v)+\cdots+\eta_2\phi_{1,1}(t,\eta,v)+\phi_{1,0}(t,\eta,v)$$
$$\vdots$$
$$\dot{\eta}_{n-1}=\eta_n^p+\eta_n^{p-1}\phi_{n-1,p-1}(t,\eta,v)+\cdots+\eta_n\phi_{n-1,1}(t,\eta,v)+\phi_{n-1,0}(t,\eta,v)$$
$$\dot{\eta}_n=v+\phi_{n,0}(t,\eta,v)$$
$$y=\eta_1, \tag{3.20}$$

called p-normal form, where $v\in\mathbb{R},\ \eta\in\mathbb{R}^n$ and $y\in\mathbb{R}$ are the system input, state and output, respectively, and $p\ge1$ is an *odd* integer. The mappings $\phi_{i,j}:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R},\ i=1,\cdots,n,\ j=0,\cdots,p-1$ are C^1, involve uncertainty and may be unknown.

The condition below, which is a natural generalization of the homogeneous growth condition (1.2), characterizes a subclass of the uncertain nonlinear systems (3.20).

Assumption 3.2. There exists a constant $C>0$ such that $\forall(t,\eta,v)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$,

$$|\phi_{i,j}(t,\eta,v)|\le C(|\eta_1|^{p-j}+\cdots+|\eta_i|^{p-j}), \tag{3.21}$$

where $j=0,\cdots,p-1$ when $i=1,\cdots,n-1$. Moreover, $j=0$ when $i=n$.

The following result is an extension of Theorem 3.1.

Theorem 3.3. Under Assumption 3.2, there is a smooth dynamic output compensator (1.3) making the uncertain system (3.20) globally asymptotically stable.

This conclusion can be proved in the spirit of Theorem 3.1, except more dedicated estimations are required when constructing recursively a robust state feedback stabilizer and a homogeneous observer. The details are omitted for the reason of space. The interested reader is referred to (Yang and Lin, 2004).

4. OUTPUT FEEDBACK STABILIZATION OF UNCERTAIN CASCADE SYSTEMS

The purpose of this section is to investigate how the robust output feedback stabilization results obtained in the previous section can be extended to a family of uncertain cascade systems of the form

$$\dot{\zeta}=F_0(t,\zeta,\eta,v)$$
$$\dot{\eta}_1=\eta_2^p+\phi_1(t,\zeta,\eta,v)$$
$$\vdots$$
$$\dot{\eta}_{n-1}=\eta_n^p+\phi_{n-1}(t,\zeta,\eta,v)$$
$$\dot{\eta}_n=v+\phi_n(t,\zeta,\eta,v)$$
$$y=\eta_1, \tag{4.1}$$

where $v \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system input and output, respectively, $\zeta \in \mathbb{R}^r$ and $\eta \in \mathbb{R}^n$ are the system states, and $p \geq 1$ is an odd integer. The functions $F_0 : \mathbb{R} \times \mathbb{R}^{n+r} \times \mathbb{R} \to \mathbb{R}^r$ and $\phi_i : \mathbb{R} \times \mathbb{R}^{n+r} \times \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, n$, are C^0 with $F_0(t, 0, 0, 0) = 0$ and $\phi_i(t, 0, \cdots, 0) = 0$.

To tackle the problem of global robust stabilization by smooth output feedback for the cascade system (4.1), we make the following assumptions:

Assumption 4.1. There is a C^2 Lyapunov function $U_0(\zeta)$, which is positive definite and proper, such that for all $(t, \zeta, \eta, v) \in \mathbb{R} \times \mathbb{R}^{n+r} \times \mathbb{R}$,

$$\frac{\partial U_0}{\partial \zeta} F_0(t, \zeta, \eta, v) \leq -||\zeta||^{p+1} + K_0 \eta_1^{p+1}, \qquad (4.2)$$

where $K_0 > 0$ is a real constant.

Assumption 4.2. There exists a constant $C > 0$ such that $\forall (t, \zeta, \eta, v) \in \mathbb{R} \times \mathbb{R}^{n+r} \times \mathbb{R}$ and $i = 1, \cdots, n$,

$$|\phi_i(t, \zeta, \eta, v)| \leq C(||\zeta||^p + |\eta_1|^p + \cdots + |\eta_i|^p). \qquad (4.3)$$

Clearly, Assumption 4.1 is a sort of ISS-like condition, while Assumption 4.2 is a natural generalization of the homogeneous growth condition (1.2). With the help of the two aforementioned conditions, we can establish the following global output feedback stabilization result for the uncertain cascade system (4.1) with uncontrollable/unobservable linearization.

Theorem 4.3. Under Assumptions 4.1-4.2, the uncertain cascade system (4.1) is globally robustly stabilizable by smooth output feedback.

Proof: The proof of this result is similar to that of Theorem 3.1. A key difference lies in the design of a partial-state (rather than full-state) feedback controller for the uncertain cascade system (4.1). For this reason, we give only a sketch of the proof with an emphasis on the difference.

As done in the proof of Theorem 3.1, we first introduce a rescaling transformation that is composed of $\zeta = \zeta$ and (3.1) for the uncertain system (4.1). Such a transformation results in

$$\begin{aligned}
\dot{\zeta} &= F_0(t, \zeta, \eta, v) \\
\dot{x}_1 &= M x_2^p + F_1(t, \zeta, x, u) \\
&\vdots \\
\dot{x}_{n-1} &= M x_n^p + F_{n-1}(t, \zeta, x, u) \\
\dot{x}_n &= M u + F_n(t, \zeta, x, u) \\
y &= x_1,
\end{aligned} \qquad (4.4)$$

where the system uncertainty satisfies the constraints:

$$\begin{aligned}
|F_1(\cdot)| &= |\phi_1(\cdot)| \leq C(||\zeta||^p + |\eta_1|^p) \leq C(||\zeta||^p + |x_1|^p), \\
|F_2(\cdot)| &= \left| \frac{\phi_2(\cdot)}{M^{\frac{1}{p}}} \right| \leq \frac{C(||\zeta||^p + |\eta_1|^p + |\eta_2|^p)}{M^{\frac{1}{p}}} \\
&\leq C M^{1-\frac{1}{p}} \left(||\zeta||^p + |x_1|^p + |x_2|^p \right), \\
&\vdots \qquad\qquad (4.5) \\
|F_n(\cdot)| &= \left| \frac{\phi_n(\cdot)}{M^{\frac{1}{p}+\cdots+\frac{1}{p^{n-1}}}} \right| \leq \frac{C(||\zeta||^p + |\eta_1|^p + \cdots + |\eta_n|^p)}{M^{\frac{1}{p}+\cdots+\frac{1}{p^{n-1}}}} \\
&\leq C M^{1-\frac{1}{p^{n-1}}} \left(||\zeta||^p + |x_1|^p + \cdots + |x_n|^p \right).
\end{aligned}$$

For the rescaled system (4.4) with the constraint (4.5), it is not difficult to see that Assumptions 4.1-4.2 imply the existence of a globally stabilizing, partial-state feedback controller $u(x_1, \cdots, x_n)$. This conclusion can be proved via the tool of adding a power integrator, as illustrated below.

Consider the Lyapunov function

$$\hat{U}_0(\zeta) = (n+6) M U_0(\zeta). \qquad (4.6)$$

Then, it follows from (4.2) that

$$\dot{\hat{U}}_0 \leq M \Big[-(n+6)||\zeta||^{p+1} + (n+6) K_0 x_1^{p+1} \Big]. \qquad (4.7)$$

Let $\xi_1 = x_1$ and choose the Lynapunov function

$$U_1(\zeta, \xi_1) = \hat{U}_0(\zeta) + \frac{1}{2}\xi_1^2.$$

Using the fact that $M \geq 1$ and (4.5), one deduces from (4.7) that

$$\begin{aligned}
\dot{U}_1 \leq\ & M \Big[-(n+6)||\zeta||^{p+1} + (n+6) K_0 x_1^{p+1} \Big] \\
& + \xi_1 \Big(M x_2^p + F_1(t, \zeta, x, u) \Big) \\
\leq\ & M \Big[-(n+6)||\zeta||^{p+1} + (n+6) K_0 \xi_1^{p+1} + \xi_1 x_2^{*p} \\
& + C|\xi_1|(||\zeta||^p + |\xi_1|^p) + \xi_1 (x_2^p - x_2^{*p}) \Big] \\
\leq\ & M \Big[-(n+5)||\zeta||^{p+1} + \xi_1 x_2^{*p} + K \xi_1^{p+1} + \xi_1 (x_2^p - x_2^{*p}) \Big],
\end{aligned}$$

where $K > 0$ is a generic constant independent of M.

Clearly, the virtual controller $x_2^* = -\beta_1 \xi_1$, with $\beta_1 = (K + n + 5)^{1/p}$ being a constant independent of M, yields

$$\dot{U}_1 \leq M \Big[-(n+5)(||\zeta||^{p+1} + \xi_1^{p+1}) + \xi_1 (x_2^p - x_2^{*p}) \Big].$$

By a similar inductive argument, at the n-th step, we conclude that there exist a set of transformations

$$\xi_i = x_i - x_i^*, \qquad x_i^* = -\beta_{i-1}\xi_{i-1}, \qquad i = 1, \cdots, n, \quad (4.8)$$

a Lyapunov function

$$U_n(\zeta, \xi_1, \cdots, \xi_n) = (n+6) M U_0(\zeta) + \frac{1}{2}(\xi_1^2 + \cdots + \xi_n^2),$$

and a partial-state feedback control law of the form

$$x_{n+1}^* = -\beta_n \xi_n = -(b_1 x_1 + \cdots + b_n x_n)^p, \qquad (4.9)$$

such that

$$\begin{aligned}
\dot{U}_n(\zeta, \xi_1, \cdots, \xi_n) \leq M \Big[& -6(||\zeta||^{p+1} + \xi_1^{p+1} + \cdots + \xi_n^{p+1}) \\
& + \xi_n (u - x_{n+1}^*) \Big],
\end{aligned} \qquad (4.10)$$

where all the parameters $\beta_1, \cdots, \beta_k$ and $b_1, \cdots, b_k > 0$ are known constants independent of M. Note that inequality (4.10) reduces to (3.5) in the absence of ζ-dynamics.

Because the states $(x_2, \cdots, x_n)$ are unmeasurable, the controller (4.9) cannot be directly implemented. To obtain an implementable controller, we design an $(n-1)$-dimensional observer for recovering $(x_2, \cdots, x_n)$ of the rescaled system (4.4). Motivated by the robust observer design in the last section, we ignore the uncertain terms $F_i(t, \zeta, x, u), i = 1, \cdots, n$, in system (4.4) and construct the dynamic output compensator

$$\begin{aligned}
\dot{\hat{z}}_2 &= M(\hat{z}_3 + L_3 L_2 x_1)^p - M L_2 (\hat{z}_2 + L_2 x_1)^p \\
&\vdots \\
\dot{\hat{z}}_n &= M u - M L_n \cdots L_2 (\hat{z}_2 + L_2 x_1)^p, \\
u &= -(b_1 x_1 + b_2 \hat{x}_2 + \cdots + b_n \hat{x}_n)^p \\
&\text{with } \hat{x}_i = \hat{z}_i + L_i \cdots L_2 x_1, \quad 2 \leq i \leq n,
\end{aligned} \qquad (4.11)$$

where $L_2, \cdots, L_n$ are the observer gains to be assigned.

The remaining part of the proof is to determine the parameters $L_2, \cdots, L_n$ as well as the rescaling factor M, which is analogous to that of Theorem 3.1 and therefore left to the reader as an exercise. In conclusion, one can prove that by suitably choosing the gain constants $L_n, L_{n-1} \cdots, L_2$ and M one-by-one, the closed-loop system (4.4)–(4.11) can be rendered globally asymptotically stable at the equilibrium $(\zeta, x, \hat{x}_2, \cdots, \hat{x}_n) = (0, 0, 0, \cdots, 0)$. □

In the remainder of this section, we discuss briefly how Theorem 3.3 for the uncertain system (3.20) in the p-normal form can be extended to the C^1 uncertain cascade system

$$\begin{aligned}
\dot{\zeta} &= F_0(t,\zeta,\eta,v), \qquad \zeta \in \mathbb{R}^r,\\
\dot{\eta}_1 &= \eta_2^p + \eta_2^{p-1}\psi_{1,p-1}(t,\zeta,\eta,v) + \cdots\\
&\quad + \eta_2\psi_{1,1}(t,\zeta,\eta,v) + \psi_{1,0}(t,\zeta,\eta,v)\\
&\vdots\\
\dot{\eta}_{n-1} &= \eta_n^p + \eta_n^{p-1}\psi_{n-1,p-1}(t,\zeta,\eta,v) + \cdots\\
&\quad + \eta_n\psi_{n-1,1}(t,\zeta,\eta,v) + \psi_{n-1,0}(t,\zeta,\eta,v)\\
\dot{\eta}_n &= v + \psi_{n,0}(t,\zeta,\eta,v)\\
y &= \eta_1. \qquad (4.12)
\end{aligned}$$

To achieve global stabilization by smooth output feedback, we assume that the ζ-subsystem of (4.12) satisfies the ISS-like condition (4.2). Moreover,

Assumption 4.4. For $i = 1, \cdots, n-1$ and $j = 0, \cdots, p-1$,

$$|\psi_{i,j}(t,\zeta,\eta,v)| \le C\Big(||\zeta||^{p-j} + \Sigma_{l=1}^{i}|\eta_l|^{p-j}\Big),$$

$$\text{and} \quad |\psi_{n,0}(t,\zeta,\eta,v)| \le C\Big(||\zeta||^{p} + \Sigma_{l=1}^{n}|\eta_l|^{p}\Big). \qquad (4.13)$$

Then, the following output feedback stabilization result holds.

Theorem 4.5. Under Assumptions 4.1 and 4.4, the uncertain cascade system (4.12) is globally asymptotically stabilizable by a smooth output feedback controller of the form (1.3).

The proof of Theorem 4.5 can be done via an argument analogous to that of Theorem 4.3, with an obvious modification. The details are omitted for the sake of space.

We conclude this section with an example that illustrates how Theorem 4.5 can be employed to solve the problem of global robust stabilization by smooth output feedback, for uncertain cascade systems which go beyond a triangular structure and contain *uncontrollable/unobservable* linearization.

Example 4.6. Consider the uncertain cascade system

$$\begin{aligned}
\dot{\zeta} &= -\zeta + \eta_1\cos(\zeta v)\\
\dot{\eta}_1 &= \eta_2^3 + \theta\eta_2\sin(\zeta\eta_1)\\
\dot{\eta}_2 &= v\\
y &= \eta_1. \qquad (4.14)
\end{aligned}$$

where θ is an unknown constant bounded by a known constant, for instance, by one.

Note that this nonlinear system has three significant features that make global output feedback stabilization of (4.14) difficult. First of all, system (4.14) is *not in a lower-triangular form* due to the first or second dynamic equation. Secondly, the linearized system of (4.14) is

$$A = \begin{bmatrix} -1 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0\\ 0\\ 1 \end{bmatrix}, \quad C = [1\ 0\ 0],$$

which is neither controllable nor observable. While the latter makes most of the output feedback design methods inapplicable to system (4.14), the former prevents an application of the output feedback control scheme developed recently (Yang and Lin, 2003) to the cascade system (4.14). Finally, the presence of the *unknown constant* θ requires that a robust output feedback control scheme be used, and therefore makes the output feedback design method proposed in (Yang and Lin, 2003) inapplicable, due to the nature of the non-robust design.

On the other hand, it is easy to see that the three-dimensional cascade system (4.14) is of the form (4.12) with $n = 2$ and $p = 3$, and satisfies the assumption (4.13). Moreover, the ISS-like inequality (4.2) holds for the ζ-subsystem of (4.14). As a matter of fact, using the Lyapunov function $U_0(\zeta) = \zeta^4$ and Lemma 6.1 yields

$$\frac{\partial U_0}{\partial \zeta}(-\zeta + \eta_1\cos(\zeta v)) \le -3\zeta^4 + 27\eta_1^4.$$

Moreover, the uncertain term $\theta\eta_2\sin(\zeta\eta_1) = \eta_2\psi_{1,1}(\zeta,\eta,v)$. Thus,

$$|\psi_{1,1}(\zeta,\eta,v)| = |\theta\sin(\zeta x_1)| \le \frac{1}{2}|\zeta|^2 + \frac{1}{2}|x_1|^2,$$

because $|\theta| \le 1$. Hence, Assumption 4.4 holds.

By Theorem 4.5, one can design a dynamic output compensator of the form (1.3) such that the closed-loop system is globally asymptotically stable. In what follows, a design procedure is given for the purpose of illustration.

First, we introduce the rescaling transformation $\zeta = \zeta,\ x_1 = \eta_1,\ x_2 = \frac{\eta_2}{M^{1/3}},\ u = \frac{v}{M^{4/3}}$, where $M \ge 1$ is a rescaling factor to be determined later. Such a transformation results in

$$\begin{aligned}
\dot{\zeta} &= -\zeta + x_1\cos(M^{4/3}\zeta u)\\
\dot{x}_1 &= Mx_2^3 + M^{1/3}\theta x_2\sin(\zeta x_1)\\
\dot{x}_2 &= Mu\\
y &= x_1. \qquad (4.15)
\end{aligned}$$

Using the tool of adding a power integrator (Lin and Qian, 2000), we can find the Lyapunov function

$$U(\zeta, x_1, \xi_2) = MU_0(\zeta) + \frac{1}{2}(x_1^2 + \xi_2^2), \qquad \xi_2 = x_2 + a_1x_1,$$

and a partial-state feedback controller $x_3^* = -(a_2\xi_2)^3 = -[a_2(x_2 + a_1x_1)]^3$, such that

$$\dot{U} \le -M[2(\zeta^4 + x_1^4 + \xi_2^4) - \xi_2(u - x_3^*)],$$

where a_1 and a_2 are positive constants independent of M.

Next, let $z_2 = x_2 - L_2x_1$ with $L_2 > 0$ being a gain constant to be assigned later. Since

$$\dot{z}_2 = Mu - L_2[Mx_2^3 + M^{1/3}\theta x_2\sin(\zeta x_1)], \qquad (4.16)$$

we design the reduced-order observer

$$\dot{\hat{z}}_2 = Mu - L_2M\hat{x}_3^3, \qquad \text{where } \hat{x}_2 = \hat{w}_2 + L_2x_1,$$

which is a copy of (4.16) without the uncertain term $M^{1/3}\theta x_2\sin(\zeta x_1)$.

Using $\hat{x}_2$ thus obtained and the certainty equivalence principle, we deduce from x_3^* that

$$u = -[a_2(\hat{x}_2 + a_1x_1)]^3 = -[a_2(\hat{w}_2 + L_2x_1 + a_1x_1)]^3. (4.17)$$

Finally, we show that the dynamic output compensator (4.16)-(4.17) globally robustly stabilizes the uncertain cascade system (4.15) $\forall|\theta| \le 1$, if L_2 and M are chosen suitably.

To this end, Let $e_2 = x_2 - \hat{x}_2 \equiv z_2 - \hat{z}_2$ be the estimate error. The error dynamics is

$$\dot{e}_2 = -L_2M(x_2^3 - \hat{x}_2^3) - L_2M^{1/3}\theta x_2\sin(\zeta x_1).$$

Choose the Lyapunov function $W = \frac{e_2^2}{2}$. Then,

$$\begin{aligned}
\dot{W} &= e_2\Big[-L_2M(x_2^3 - \hat{x}_2^3) - L_2M^{1/3}\theta x_2\sin(\zeta x_1)\Big]\\
&\le M\Big(-\frac{L_2}{4}e_2^4 + L_2M^{-2/3}|\zeta x_1x_2e_2|\Big), \qquad \forall\ |\theta| \le 1.
\end{aligned}$$

Selecting $M \ge L_2^{3/2}$ yields

$$\begin{aligned}
\dot{W} &\le M\Big(-\frac{L_2}{4}e_2^4 + |\zeta x_1x_2e_2|\Big)\\
&\le M\Big[-\frac{L_2}{4}e_2^4 + \frac{1}{2}(\zeta^4 + x_1^4 + \xi_2^4) + Ke_2^4\Big],
\end{aligned}$$

where $K > 0$ is a constant independent of M.

Now, consider $V(\zeta, x_1, \xi_2, e_2) = U(\zeta, x_1, \xi_2) + W(e_2)$ for the closed-loop system (4.15)-(4.16)-(4.17). Using Lemmas 6.1-6.5, it is not difficult to prove that

$$\dot{V} \leq M\Big[-(\frac{L_2}{4} - K)e_2^4 - (\zeta^4 + x_1^4 + \xi_2^4)\Big].$$

In view of $M \geq L_2^{3/2}$ and $M > 1$, it is clear that the choices $L_2 = 4(K+1)$ and $M = \max(L_2^{3/2}, 1)$ result in

$$\dot{V} \leq -M(\zeta^4 + x_1^4 + \xi_2^4 + e_2^4),$$

which implies the closed-loop system (4.15)-(4.16)-(4.17) is globally asymptotically stable $\forall \theta \in [-1, 1]$.

The aforementioned design procedure leads to, for instance, the dynamic output compensator

$$\begin{aligned} \dot{\hat{z}}_2 &= 0.4v - 80(\hat{z}_2 + 24x_1)^3 \\ v &= -34[24(\hat{w}_2 + 11.4x_1)]^3, \end{aligned} \tag{4.18}$$

that does the job.

The simulation shown in Fig. 1 demonstrates global asymptotic stability of the closed-loop system (4.14)-(4.18).

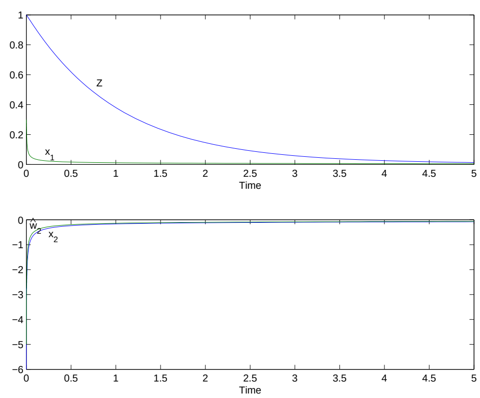

Fig. 1 Transient responses of the closed-loop system (4.14)-(4.18) with $\theta = 1$ and the initial condition $(\zeta, x_1, x_2, \hat{z}_2) = (1, 0.3, -6, -5)$, where $\hat{w}_2 = \hat{z}_2, z = \zeta$.

5. CONCLUSION

This paper has proved that under an appropriate homogeneous growth condition, global robust stabilization by smooth output feedback can be achieved for a family of uncertain nonlinear systems whose linearization is uncontrollable and unobservable. A robust output feedback design approach has been developed based on the idea of a non-separation principle design (Qian and Lin, 2002*a*) and a rescaling technique, enabling one to recursively construct a robust state feedback controller and a homogeneous observer that does not depend on the uncertainty of the system. The main results of this paper have incorporated and generalized the robust output feedback stabilization theorem in (Qian and Lin, 2002*a*), where global exponential stabilization was shown to be possible for a family of uncertain systems with controllable/observable linearization under a linear growth condition.

6. APPENDIX

This section collects several useful lemmas that play a key role in deriving the main results of this paper.

Lemma 6.1. Given positive real numbers x, y, m, n, a, b, the following inequality holds:

$$ax^m y^n \leq bx^{m+n} + \frac{n}{m+n}(\frac{m+n}{m})^{-\frac{m}{n}} a^{\frac{m+n}{n}} b^{-\frac{m}{n}} y^{m+n}.$$

Lemma 6.2. Given positive real numbers x, y, m, n, a, b, the following inequality holds:

$$abx^m y^n \leq \frac{m}{m+n} a^{\frac{m+n}{m}} x^{m+n} + \frac{n}{m+n} b^{\frac{m+n}{n}} y^{m+n}.$$

Lemmas 6.1-6.2 can be proved by the Young's Inequality.

Lemma 6.3. Let $x_1, \cdots, x_n, p > 0$ be real numbers. Then,

$$(x_1 + \cdots + x_n)^p \leq \max(n^{p-1}, 1)(x_1^p + \cdots + x_n^p).$$

Lemma 6.4. Let x and y be any real numbers and $p > 0$ be an odd integer. Then, the following inequality holds:

$$-(x-y)(x^p - y^p) \leq -\frac{1}{2^{p-1}}(x-y)^{p+1}.$$

Lemma 6.5. For all $x, y \in \mathbb{R}$ and any odd positive integer p, the following inequality holds:

$$|x^p - y^p| \leq p|x-y|(x^{p-1} + y^{p-1}).$$

The proofs of Lemmas 6.3—6.5 are not difficult and hence left to the reader as an exercise.

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