ON FLATNESS NECESSARY AND SUFFICIENT CONDITIONS

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Abstract: We give a characterization of differentially flat nonlinear systems in implicit representation, where the input variables are eliminated. Lie-Bäcklund isomorphisms associated to a flat system, called *trivializations*, can be locally characterized in terms of matrices polynomial with respect to $\frac{d}{dt}$. Such polynomial matrices are useful to compute the ideal of differential forms generated by the differentials of all possible trivializations. We introduce the notion of a *strongly closed ideal* of differential forms, and prove that flatness is equivalent to the strong closedness of the latter ideal.

Keywords: Nonlinear system, implicit system, manifold of jets of infinite order, polynomial matrices, ideals, differential forms, differential flatness.

1. INTRODUCTION

Differential flatness, or more shortly, flatness, has been introduced more than ten years ago in Martin (1992); Fliess et al. (1992).

Roughly speaking, given a nonlinear system

$$\dot{x} = f(x, u) \qquad (1)$$

where $x = (x_1, \ldots, x_n)$ is the state and $u = (u_1, \ldots, u_m)$ the control vector, $m \leq n$, the system (1) is said to be flat if and only if there exists a vector $y = (y_1, \ldots, y_m)$ such that:
(i) y and its successive time derivatives $\dot{y}, \ddot{y}, \ldots,$ are independent,
(ii) y is a function of x, u and a finite number of time derivatives of the components of u,
(iii) x and u can be expressed as functions of the components of y and a finite number of their derivatives: $x = \varphi(y, \dot{y}, \ldots, y^{(\alpha)})$, $u = \psi(y, \dot{y}, \ldots, y^{(\alpha+1)})$, for some multi-integer $\alpha = (\alpha_1, \ldots, \alpha_m)$, and with the notation $y^{(\alpha)} = (\frac{d^{\alpha_1} y_1}{dt^{\alpha_1}}, \ldots, \frac{d^{\alpha_m} y_m}{dt^{\alpha_m}})$.

The vector y is called a *flat output*.

This concept has inspired an important literature (see e.g. Martin et al. (1997) for a survey). Various formalisms have been introduced: finite dimensional differential geometric approaches (Charlet et al. (1991); Franch (1999); Shadwick (1990); Sluis (1993)), differential algebra and related approaches (Fliess et al. (1995); Aranda-Bricaire et al. (1995); Jakubczyk (1993)), infinite dimensional differential geometry of jets and prolongations (Fliess et al. (1999); Nieuwstadt et al. (1998); Pomet (1993); Pereira da Silva and Filho (2001); Rathinam and Murray (1998)).

Flatness may be traced back to Hilbert (1912) and Cartan (1914). Moreover, using the present definition, with, in place of (1), the set of $n - m$ implicit equations (2) where the control variables u are eliminated, this property may be seen as a generalization in the framework of manifolds of jets of infinite order of the *uniformization of analytic functions* of Hilbert 's 22nd problem (Hilbert (1901)), solved by Poincaré (1907). This problem consists, roughly speaking, given a set of complex polynomial equations in one complex variable, in finding an open dense subset D of the complex plane $\mathbb{C}$ and a holomorphic function s from D to $\mathbb{C}$ such that s is surjective and $s(p)$ identically satisfies the given equations for all values of the "parameter" $p \in D$. Here, $\mathbb{C}$ is replaced by a (real) manifold of jets of infinite order, a flat output $y_1, \ldots, y_m$ plays the role of the parameter p and s is the associated Lie-Bäcklund isomorphism $s = (\varphi, \psi, \dot{\psi}, \ddot{\psi}, \ldots)$ with φ and ψ defined above.

In the framework of linear finite or infinite dimensional systems, the notions of flatness and *parametrization* coincide as remarked by Pommaret (2001); Pommaret and Quadrat (1999), and in the behavioral approach (Polderman and J.C.Willems (1997)), flat outputs correspond to *latent variables* of *observable image representations* (Trentelman (2004). See also Fliess (1992) for a module theoretic interpretation of the behavioral approach).

The characterization of differentially flat systems has aroused many contributions: Aranda-Bricaire et al. (1995); Charlet et al. (1991); Chetverikov (2001); Franch (1999); Jakubczyk (1993); Martin and Rouchon (1993); Pereira da Silva (2000); Pomet (1997); Rathinam and Murray (1998); Rouchon (1994); Shadwick (1990); Sluis (1993). For general necessary and sufficient conditions, see e.g. Aranda-Bricaire et al. (1995); Chetverikov (2001); Pereira da Silva (2000).

We adopt here the formalism of manifolds of jets of infinite order (Fliess et al. (1999); Krasil'shchik et al. (1986); Pomet (1993); Zharinov (1992)) and, as previously mentioned, we consider implicit systems (2) obtained from (1) by eliminating the input vector u. We adapt the notions of Lie-Bäcklund equivalence and Lie-Bäcklund isomorphism in this context and show, after restricting to the category of meromorphic functions, that flatness is naturally described in terms of polynomial matrices and differential forms.

Though our results show some parallelism with those of Aranda-Bricaire et al. (1995); Chetverikov (2001), they exploit different ideas that may be seen as an extension to nonlinear systems of Lévine and Nguyen (2003) and they provide flatness conditions that are invariant by endogeneous dynamic feedback.

The paper is organized as follows: Section 2 is devoted to the basic description of implicit control systems on manifolds of jets of infinite order. In Section 3, we extend the notions of Lie-Bäcklund equivalence and Lie-Bäcklund isomorphism to the implicit system framework and, in Section 4, of flat systems. The necessary and sufficient conditions for flatness are stated in Theorem 9 of Section 5. Section 6 is then devoted to an example. In the sequel, all the proofs are omitted due to a lack of space.

2. IMPLICIT CONTROL SYSTEMS ON MANIFOLDS OF JETS OF INFINITE ORDER

Given an infinitely differentiable manifold X of dimension n, we denote its tangent space at $x \in X$ by $\mathrm{T}_x X$, and its tangent bundle by $\mathrm{T}X$. Let F belong to $C^\infty(\mathrm{T}X; \mathbb{R}^{n-m})$, the set of C^∞ mappings from $\mathrm{T}X$ to $\mathbb{R}^{n-m}$. We consider an underdetermined implicit system of the form

$$F(x, \dot{x}) = 0 \qquad (2)$$

regular in the sense that $\mathrm{rank}\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m$ in a suitable open subset of $\mathrm{T}X$.

According to the implicit function theorem, any explicit system (1) with $x \in X$, $(x, f(x,u)) \in \mathrm{T}_x X$ for every u in an open subset U of $\mathbb{R}^m$, and $\mathrm{rank}\left(\frac{\partial f}{\partial u}\right) = m$ in a suitable open subset of $X \times U$, can be locally transformed into (2), and conversely.

A vector field f that depends, for every $x \in X$, on m independent variables $u \in \mathbb{R}^m$ in a C^∞ way with $\mathrm{rank}\left(\frac{\partial f}{\partial u}\right) = m$ in a suitable open subset of $X \times \mathbb{R}^m$, satisfying $F(x, f(x,u)) = 0$ for every $u \in U$, is called *compatible* with (2).

In Fliess et al. (1999) (see also Pomet (1993) where a similar approach has been developed independently), infinite systems of coordinates $(x, \overline{u}) = (x, u, \dot{u}, \ldots)$ have been introduced to deal with prolonged vector fields $\overline{f}(x, \overline{u}) = \sum_{i=1}^n f_i(x,u) \frac{\partial}{\partial x_i} + \sum_{j=1}^m \sum_{k\geq 0} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}}$, the original system being in explicit form (1).

Here, we adopt an external description of the prolonged manifold containing the solutions of (2): we consider the infinite dimensional manifold $\mathfrak{X}$ defined by $\mathfrak{X} \stackrel{\text{def}}{=} X \times \mathbb{R}^n_\infty \stackrel{\text{def}}{=} X \times \mathbb{R}^n \times \mathbb{R}^n \times \ldots$ made of an infinite (but countable) number of copies of $\mathbb{R}^n$, endowed with the product topology on $X \times \mathbb{R}^n_\infty$. We assume that we are given the global infinite set of coordinates of $\mathfrak{X}$:

$$\overline{x} = (x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, \ddot{x}_1, \ldots, \ddot{x}_n, \ldots, \ldots, x_1^{(k)}, \ldots, x_n^{(k)}, \ldots). \qquad (3)$$

Definition 1. We say that a function φ from $\mathfrak{X}$ to $\mathbb{R}$ is *continuous* (resp. *differentiable*) if φ depends only on a finite (but otherwise arbitrary) number of variables and is continuous (resp. differentiable) with respect to these variables.

C^∞ or analytic or meromorphic functions from $\mathfrak{X}$ to $\mathbb{R}$ are then defined as in the usual finite dimensional case since they only depend on a finite number of variables.

We endow $\mathfrak{X}$ with the so-called trivial Cartan vector field (Krasil'shchik et al. (1986); Zharinov (1992))

$$\tau_{\mathfrak{X}} = \sum_{i=1}^n \sum_{j\geq 0} x_i^{(j+1)} \frac{\partial}{\partial x_i^{(j)}}. \qquad (4)$$

We also denote by $L_{\tau_{\mathfrak{X}}} \varphi = \sum_{i=1}^n \sum_{j\geq 0} x_i^{(j+1)} \frac{\partial \varphi}{\partial x_i^{(j)}} = \frac{d\varphi}{dt}$ the Lie derivative of a differentiable function φ along $\tau_{\mathfrak{X}}$ and $L^k_{\tau_{\mathfrak{X}}} \varphi$ its kth iterate. Thus $x_i^{(k)} = \frac{d^k x_i}{dt^k} =$

$L^k_{\tau_{\mathfrak{X}}} x_i$ for every $i = 1, \ldots, n$ and $k \geq 1$, with the convention $x_i^{(0)} = x_i$.

Since $\frac{d}{dt} x_i^{(j)} \stackrel{\text{def}}{=} \dot{x}_i^{(j)} = x_i^{(j+1)}$, the Cartan vector field acts on coordinates as a shift to the right. $\mathfrak{X}$ is thus called *manifold of jets of infinite order*.

From now on, $\overline{x}\ \overline{y}, \ldots$ stand for the sequence of jets of infinite order of x, y,...

Note that the implicit representation (2), as opposed to (1), has the obvious advantage to be invariant by endogeneous dynamic feedback (see Fliess et al. (1999) for a precise definition).

A regular implicit control system is thus defined as a triple $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ *with* $\mathfrak{X} = X \times \mathbb{R}^n_\infty$, $\tau_{\mathfrak{X}}$ *its associated trivial Cartan field, and* $F \in C^\infty(\mathrm{T}X; \mathbb{R}^{n-m})$ *satisfying* $\mathrm{rank}\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m$ *in a suitable open subset of* $\mathrm{T}X$.

3. LIE-BÄCKLUND EQUIVALENCE FOR IMPLICIT SYSTEMS

We now slightly adapt the notion of Lie-Bäcklund equivalence[1] of Fliess et al. (1999) in our implicit control system context:

Let us consider two regular implicit control systems $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$, with $\mathfrak{X} = X \times \mathbb{R}^n_\infty$, $\dim X = n$ and $\mathrm{rank}\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m$, and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$, with $\mathfrak{Y} = Y \times \mathbb{R}^p_\infty$, $\dim Y = p$, $\tau_{\mathfrak{Y}}$ its trivial Cartan field, and $\mathrm{rank}\left(\frac{\partial G}{\partial \dot{y}}\right) = p - q$.

Set $\mathfrak{X}_0 = \{\overline{x} \in \mathfrak{X} | L^k_{\tau_{\mathfrak{X}}} F(\overline{x}) = 0,\ \forall k \geq 0\}$ and $\mathfrak{Y}_0 = \{\overline{y} \in \mathfrak{Y} | L^k_{\tau_{\mathfrak{Y}}} G(\overline{y}) = 0,\ \forall k \geq 0\}$. They are endowed with the topologies and differentiable structures induced by $\mathfrak{X}$ and $\mathfrak{Y}$ respectively.

Definition 2. We say that the regular implicit control systems $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ are *Lie-Bäcklund equivalent* (or shortly L-B equivalent) at the pair of points $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathfrak{Y}_0$ if and only if

(i) there exist neighborhoods $\mathcal{X}_0$ and $\mathcal{Y}_0$ of $\overline{x}_0$ in $\mathfrak{X}_0$ and $\overline{y}_0$ in $\mathfrak{Y}_0$ respectively and a one-to-one mapping $\Phi = (\varphi_0, \varphi_1, \ldots) \in C^\infty(\mathcal{Y}_0; \mathcal{X}_0)$ satisfying $\Phi(\overline{y}_0) = \overline{x}_0$ and such that the trivial Cartan fields are Φ-related, namely $\Phi_* \tau_{\mathfrak{Y}} = \tau_{\mathfrak{X}}$;

(ii) there exists $\Psi \in C^\infty(\mathcal{X}_0; \mathcal{Y}_0)$, one-to-one, $\Psi = (\psi_0, \psi_1, \ldots)$, such that $\Psi(\overline{x}_0) = \overline{y}_0$ and $\Psi_* \tau_{\mathfrak{X}} = \tau_{\mathfrak{Y}}$.

The mappings Φ and Ψ are called *mutually inverse Lie-Bäcklund isomorphisms* at $(\overline{x}_0, \overline{y}_0)$.

The two systems $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ are called *locally L-B equivalent* if they are L-B equivalent at every pair $(\overline{x}, \Psi(\overline{x})) = (\Phi(\overline{y}), \overline{y})$ of an open dense subset $\mathcal{Z}$ of $\mathfrak{X}_0 \times \mathfrak{Y}_0$, with Φ and Ψ mutually inverse Lie-Bäcklund isomorphisms on $\mathcal{Z}$.

It is immediately seen that, in suitable neighborhoods, two systems $\dot{x} = f(x, u)$ and $\dot{y} = g(y, v)$ with f compatible with $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ and g compatible with $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$, are locally *differentially equivalent* in the sense of Fliess et al. (1999) (or locally L-B equivalent as proposed in footnote [1]), if and only if $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ are locally L-B equivalent in the sense of Definition 2 in a suitable open dense subset.

Clearly, local L-B equivalence preserves equilibrium points, namely points $\tilde{y}$ (resp. $\tilde{x}$) such that $G(\tilde{y}, 0) = 0$ (resp. $F(\tilde{x}, 0) = 0$) and coranks ($m = q$, easy adaptation of Fliess et al. (1999)).

3.1 1-forms

Let us introduce a basis of the tangent space $\mathrm{T}_{\overline{x}}\mathfrak{X}$ of $\mathfrak{X}$ at a point $\overline{x} \in \mathfrak{X}$ consisting of the set of vectors $\{\frac{\partial}{\partial x_i^{(j)}} | i = 1, \ldots, n,\ j \geq 0\}$. A basis of the cotangent space $\mathrm{T}^*_{\overline{x}}\mathfrak{X}$ at $\overline{x}$ is therefore given by $\{dx_i^{(j)} | i = 1, \ldots, n,\ j \geq 0\}$ with $< dx_i^{(j)}, \frac{\partial}{\partial x_k^{(l)}} >= \delta_{i,k}\delta_{j,l}$, $\delta_{i,k}$ being the Kronecker symbol.

The differential of F is thus given, in matrix notations, by

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial \dot{x}} d\dot{x}. \tag{5}$$

Note that the shift property of $\frac{d}{dt}$ on coordinates extends to differentials: $\frac{d}{dt} dx = d\dot{x} = d \frac{d}{dt} x$, i.e. $\frac{d}{dt}$ commutes with d.

Since a smooth function depends on a finite number of variables, its differential contains only a finite number of non zero terms. Accordingly, we define a 1-form on $\mathfrak{X}$ as a *finite* linear combination of the $dx_i^{(j)}$'s, with coefficients in $C^\infty(\mathfrak{X}; \mathbb{R})$ or, equivalently as a local C^∞ section of $\mathrm{T}^*\mathfrak{X}$. The set of 1-forms is noted $\Lambda^1(\mathfrak{X})$.

If Φ is a C^∞ mapping from $\mathfrak{Y}$ to $\mathfrak{X}$, the definition of the (backward) image by Φ of a 1-form is the same as in the finite dimensional context.

4. FLATNESS

First recall from Fliess et al. (1999) that a system in explicit form is flat if and only if it is L-B equivalent to a trivial system. The reader may easily check that this definition is just a concise restatement of the definiton given in the Introduction. In our implicit context, it reads:

[1] The terminology introduced by Fliess et al. (1999) needs, in the author's opinion, to be unified: *differential equivalence* (corresponding to *endogeneous transformations* and Φ*-related Cartan fields*) and Lie-Bäcklund equivalence (including time scalings into the previous endogeneous transformations by replacing Cartan fields by Cartan distributions). We propose to use *Lie-Bäcklund* (resp. *orbital Lie-Bäcklund) equivalence* in place of *differential* (resp. *Lie-Bäcklund) equivalence* and *Lie-Bäcklund* (resp. *orbital Lie-Bäcklund) isomorphisms* in place of *endogeneous transformations* (resp. *Lie-Bäcklund isomorphisms*).

Definition 3. The implicit system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is *flat* at $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathbb{R}^m_\infty$ if and only if it is L-B equivalent at $(\overline{x}_0, \overline{y}_0)$ to the trivial implicit system $(\mathbb{R}^m_\infty, \tau_{\mathbb{R}^m_\infty}, 0)$. In this case, the inverse L-B isomorphisms Φ and Ψ are called *inverse trivializations*, or *uniformizations* (in reference to Hilbert's 22nd problem).

Theorem 4. The system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is flat at $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathbb{R}^m_\infty$ if and only if there exists a local smooth invertible mapping Φ from $\mathbb{R}^m_\infty$ to $\mathfrak{X}_0$, with smooth inverse, satisfying $\Phi(\overline{y}_0) = \overline{x}_0$, and such that

$$\Phi^* dF = 0. \qquad (6)$$

5. FLATNESS NECESSARY AND SUFFICIENT CONDITIONS

We now analyze condition (6) in more details. In order to use the algebraic properties of polynomial matrices and of modules over a principal ideal ring of polynomials, we make the (mild) restriction that F is meromorphic on $\mathrm{T}X$ and that the inverse trivializations Φ and Ψ of definition 3 belong to the class of meromorphic functions.

We introduce the following matrices polynomial with respect to the differential operator $\frac{d}{dt}$ (we use indifferently $\frac{d}{dt}$ for $L_{\tau_{\mathfrak{X}}}$ or $L_{\tau_{\mathbb{R}^m_\infty}}$, the context being unambiguous):

$$P(F) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}}\frac{d}{dt}, \quad P(\varphi_0) = \sum_{j \geq 0} \frac{\partial \varphi_0}{\partial y^{(j)}} \frac{d^j}{dt^j} \qquad (7)$$

with $P(F)$ (resp. $P(\varphi_0)$) of size $(n-m) \times n$ (resp. $n \times m$), (6) reads:

$$\Phi^* dF_{\big|\overline{y}} = P(F)_{\big|\Phi(\overline{y})} P(\varphi_0)_{\big|\overline{y}}\, dy = 0 \qquad (8)$$

or equivalently

$$\Phi^* dF_{\big|\Psi(\overline{x})} = P(F)_{\big|\overline{x}} P(\varphi_0)_{\big|\Psi(\overline{x})} dy = 0. \qquad (9)$$

Clearly, the entries of these matrices are polynomials of the differential operator $\frac{d}{dt}$ with meromorphic coefficients from $\mathfrak{X}$ to $\mathbb{R}$. We denote by $\mathfrak{K}$ the field of meromorphic functions from $\mathfrak{X}$ to $\mathbb{R}$ and by $\mathfrak{K}[\frac{d}{dt}]$ the principal ideal ring of polynomials of $\frac{d}{dt}$ with coefficients in $\mathfrak{K}$.

Note that $\mathfrak{K}[\frac{d}{dt}]$ is non commutative, even if $n = 1$: for every $a \in \mathfrak{K}$, $a \neq 0$, we have $\left(\frac{d}{dt} \cdot x - x \cdot \frac{d}{dt}\right)(a) = \dot{x}a + x\dot{a} - x\dot{a} = \dot{x}a \neq 0$, or $\frac{d}{dt} \cdot x - x \cdot \frac{d}{dt} = \dot{x}$.

For $p, q \in \mathbb{N}$, let us denote by $\mathcal{M}_{p,q}[\frac{d}{dt}]$ the module of $p \times q$ matrices over $\mathfrak{K}[\frac{d}{dt}]$ (for modules over a non commutative ring see e.g. Cohn (1985)). Recall that, for any $p \in \mathbb{N}$, the inverse of a square invertible matrix of $\mathcal{M}_{p,p}[\frac{d}{dt}]$ is not in general in $\mathcal{M}_{p,p}[\frac{d}{dt}]$. Matrices whose inverse belong to $\mathcal{M}_{p,p}[\frac{d}{dt}]$ are called *unimodular matrices* and their set is denoted by $\mathcal{U}_p[\frac{d}{dt}]$. Here, $P(F) \in \mathcal{M}_{n-m,n}[\frac{d}{dt}]$. It admits a *Smith decomposition* (or diagonal reduction), given by

$$VP(F)U = \left(\Delta, 0_{n-m,m}\right) \qquad (10)$$

with $0_{n-m,m}$ the $(n-m) \times m$ matrix whose entries are all zeros, $V \in \mathcal{U}_{n-m}[\frac{d}{dt}]$, $U \in \mathcal{U}_n[\frac{d}{dt}]$ and $\Delta \in \mathcal{M}_{n-m,n-m}[\frac{d}{dt}]$ a diagonal matrix whose entries $d_{i,i}$ divide $d_{j,j}$ for all $0 \leq i \leq j \leq n-m$. Moreover, Δ is unique (see Cohn (1985)).

Definition 5. A matrix $M \in \mathcal{M}_{p,q}[\frac{d}{dt}]$ is said *hyper-regular* if and only if its Smith decomposition leads to $\left(I_p, 0_{p,q-p}\right)$ if $p < q$, to I_p if $p = q$, and to $\begin{pmatrix} I_q \\ 0_{p-q,q} \end{pmatrix}$ if $p > q$.

Note that a square matrix $M \in \mathcal{M}_{p,p}[\frac{d}{dt}]$ is hyper-regular if and only if it is unimodular.

According to Fliess (1990), given a point $\overline{x} \in \mathfrak{X}$ and its projection x on the original manifold X, we denote by $\xi = (\xi_1, \ldots, \xi_n)$ a basis of the tangent space $\mathrm{T}_x X$, by $[\xi]$ the $\mathfrak{K}[\frac{d}{dt}]$-module generated by ξ, by $[P(F)\xi]$ the submodule generated by the lines of $P(F)\xi$ and by $\mathcal{M}$ the quotient module $[\xi]/[P(F)\xi]$, called the *variational module* of (2) at $\overline{x}$. It can be proved that, if system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is flat at $(\overline{x}_0, \overline{y}_0)$ then its variational module $\mathcal{M}$ is free in a neighborhood of $\overline{x}_0$ (and thus controllable in the sense of Fliess) and $P(F)$ is hyper-regular.

5.1 Algebraic characterization of the differential of a trivialization

From now on, we assume that $P(F)$ is hyper-regular in a neighborhood of $\overline{x}_0$. In other words, there exist V and U such that

$$VP(F)U = \left(I_m, 0_{n-m,m}\right). \qquad (11)$$

U and V satisfying (11) are indeed non unique. We say that $U \in \mathsf{R-Smith}\,(P(F))$ and $V \in \mathsf{L-Smith}\,(P(F))$ if they are such that $VP(F)U = (I_m, 0)$.

Accordingly, if $M \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ is hyper-regular with $m \leq n$, we say that $V \in \mathsf{L-Smith}\,(M)$ and $W \in \mathsf{R-Smith}\,(M)$ if $V \in \mathcal{U}_n[\frac{d}{dt}]$ and $W \in \mathcal{U}_m[\frac{d}{dt}]$ satisfy $VMW = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$.

We first solve the matrix equation:

$$P(F)\Theta = 0 \qquad (12)$$

where the entries of $\Theta \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ are not supposed to be gradients of some function φ_0.

Lemma 6. The set of hyper-regular matrices $\Theta \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ satisfying (12) is nonempty and given by

$$\Theta = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} W \qquad (13)$$

with $U \in \mathsf{R-Smith}(P(F))$ and $W \in \mathfrak{U}_m[\frac{d}{dt}]$ arbitrary.

We introduce the notation:

$$\hat{U} = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix}. \quad (14)$$

Lemma 7. For every $Q \in \mathsf{L-Smith}\left(\hat{U}\right)$ there exists $Z \in \mathfrak{U}_m[\frac{d}{dt}]$ such that

$$Q\Theta = \begin{pmatrix} I_m \\ 0_{n-m,m} \end{pmatrix} Z. \quad (15)$$

Moreover, for every $Q \in \mathsf{L-Smith}\left(\hat{U}\right)$, the submatrix $\hat{Q} = \left(0_{n-m,m}, I_{n-m}\right) Q$ is equivalent to $P(F)$ ($\exists L \in \mathfrak{U}_{n-m}[\frac{d}{dt}]$ such that $P(F) = L\hat{Q}$).

5.2 Integrability

Let us denote by $Q_{i,j} = \sum_{k\geq 0} Q_{i,j}^k \frac{d^k}{dt^k}$ the (i,j)-th polynomial entry of $Q \in \mathsf{L-Smith}\left(\hat{U}\right)$. We also denote by ω_i the 1-form

$$\omega_i = \left(\sum_{j=1}^{n}\sum_{k\geq 0} Q_{i,j}^k(\overline{x})dx_j^{(k)}\right)_{\big|\mathfrak{X}_0}, \; i = 1,\ldots,m. \quad (16)$$

Since $\hat{Q}$ is hyper-regular, the forms $\omega_1,\ldots,\omega_m$ are independent by construction.

Let us also recall that, if $\tau_1,\ldots,\tau_r$ are given independent 1-forms in $\Lambda^1(\mathfrak{X}_0)$, the $\mathfrak{K}[\frac{d}{dt}]$-ideal $\mathfrak{T}$ generated by $\tau_1,\ldots,\tau_r$, for an arbitrary integer r, is the set of all combinations with coefficients in $\mathfrak{K}[\frac{d}{dt}]$ of forms $\eta\wedge\tau_i$ with η an arbitrary form on $\mathfrak{X}_0$ of arbitrary degree and $i = 1,\ldots,r$.

Definition 8. We say that a $\mathfrak{K}[\frac{d}{dt}]$-ideal $\mathfrak{T}$ generated by $\tau_1,\ldots,\tau_r$ is *strongly closed* if there exists a matrix $M \in \mathfrak{U}_m[\frac{d}{dt}]$ such that $d\tau = -M^{-1}dM \wedge \tau$.

This definition is indeed independent of the choice of generators.

Theorem 9. A necessary and sufficient condition for system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ to be flat at $(\overline{x}_0, \overline{y}_0)$ is that there exist $U \in \mathsf{R-Smith}(P(F))$ and $Q \in \mathsf{L-Smith}\left(\hat{U}\right)$, with $\hat{U}$ given by (14), such that the $\mathfrak{K}[\frac{d}{dt}]$-ideal Ω generated by the 1-forms $\omega_1,\ldots,\omega_m$ defined by (16) is strongly closed in a neighborhood of $\overline{x}_0$ in $\mathfrak{X}_0$.

6. EXAMPLE

Let us consider the 3 dimensional system with 2 inputs:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= f(u_1,u_2). \end{aligned} \quad (17)$$

with $f_i' = \frac{\partial f}{\partial u_i} \not\equiv 0$, $i = 1,2$ and $\dot{f}_i' = \frac{df_i'}{dt} \not\equiv 0$, $i = 1,2$.

In implicit form, (17) reads

$$\dot{x}_3 - f(\dot{x}_1,\dot{x}_2) = 0 \quad (18)$$

and its variational system is given by

$$\left(-f_1'\frac{d}{dt}, -f_2'\frac{d}{dt}, \frac{d}{dt}\right)\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = 0 \quad (19)$$

Proposition 10. System (17), or equivalently (18), is flat if f has one of the three following forms:

(a) $f(u_1,u_2) = a(u_1)u_2 + b(u_1)$, with a and b meromorphic functions and $a \neq 0$. In this case, a minimal flat output is given by $y_1 = x_1$, $y_2 = x_3 - (a(\dot{x}_1)x_2 + b(\dot{x}_1))$.

(b) $f(u_1,u_2) = a(u_2)u_1 + b(u_2)$, with a and b meromorphic functions and $a \neq 0$. In this case, a minimal flat output is given by $y_1 = x_2$, $y_2 = x_3 - (a(\dot{x}_2)x_1 + b(\dot{x}_2))$.

(c) f satisfies the partial differential equation

$$\frac{\partial^2 f}{\partial u_1^2}\left(\frac{\partial f}{\partial u_2}\right)^2 - 2\frac{\partial^2 f}{\partial u_1\partial u_2}\frac{\partial f}{\partial u_1}\frac{\partial f}{\partial u_2} + \frac{\partial^2 f}{\partial u_2^2}\left(\frac{\partial f}{\partial u_1}\right)^2 = 0. \quad (20)$$

In this case, there exists a meromorphic function g such that $g(\dot{x}_3) = \frac{f_1'}{f_2'}(\dot{x}_1,\dot{x}_2)$ for all $(\dot{x}_1,\dot{x}_2)$ satisfying $f(\dot{x}_1,\dot{x}_2) = \dot{x}_3$ and a minimal flat output is given by

$$y_1 = g(\dot{x}_3)x_1 + x_2, \quad y_2 = x_3.$$

An example of solution of (20) is given by

$$f(u_1,u_2) = a\left(\frac{\alpha_1 u_1 + \alpha_2 u_2}{\gamma + \beta_1 u_1 + \beta_2 u_2}\right) \quad (21)$$

where a is any non constant meromorphic function, $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma$ are arbitrary real numbers such that $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, and $g(\dot{x}_3) = \frac{\alpha_1 - \beta_1 a^{-1}(\dot{x}_3)}{\alpha_2 - \beta_2 a^{-1}(\dot{x}_3)}$.

Remark 11. Flat outputs for particular cases of this example have been found in Martin et al. (1997) in the case $f(u_1,u_2) = u_1u_2$, belonging to classes (a) and (b), and in Chetverikov (2001) in the case $f(u_1,u_2) = \sin\frac{u_1}{u_2}$, belonging to class (c).

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REFERENCES

E. Aranda-Bricaire, C.H. Moog, and J.-B. Pomet. A linear algebraic framework for dynamic feedback linearization. *IEEE Trans. Automat. Contr.*, 40(1): 127–132, 1995.

É. Cartan. Sur l'équivalence absolue de certains systèmes d'équations différentielles et sur certaines familles de courbes. *Bull. Soc. Math. France*, 42: 12–48, 1914. in Oeuvres Complètes, part II, vol 2, pages 1133–1168, CNRS, Paris, 1984.

B. Charlet, J. Lévine, and R. Marino. Sufficient conditions for dynamic state feedback linearization. *SIAM J. Control and Optimization*, 29(1):38–57, 1991.

V.N. Chetverikov. New flatness conditions for control systems. In *Proceedings of NOLCOS'01, St. Petersburg*, pages 168–173, 2001.

P.M. Cohn. *Free Rings and Their Relations*. Academic Press, London, 1985.

M. Fliess. Some basic structural properties of generalized linear systems. *Systems & Control Letters*, 15: 391–396, 1990.

M. Fliess. A remark on Willems' trajectory characterization of linear controllability. *Systems & Control Letters*, 19:43–45, 1992.

M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Sur les systèmes non linéaires différentiellement plats. *C.R. Acad. Sci. Paris*, I–315:619–624, 1992.

M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and examples. *Int. J. Control*, 61(6):1327–1361, 1995.

M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE Trans. Automat. Contr.*, 44(5):922–937, 1999.

J. Franch. *Flatness, Tangent Systems and Flat Outputs*. PhD thesis, Universitat Politcnica de Catalunya Jordi Girona, 1999.

D. Hilbert. Mathematische probleme. *Archiv für Mathematik und Physik*, 1:44–63 and 213–237, 1901. Also in Gesammelte Abhandlungen, Vol.3, pages 290–329, Chelsea, New York, 1965.

D. Hilbert. Über den Begriff der Klasse von Differentialgleichungen. *Math. Ann.*, 73:95–108, 1912. Also in Gesammelte Abhandlungen, Vol. 3, pages 81–93, Chelsea, New York, 1965.

B. Jakubczyk. Invariants of dynamic feedback and free systems. In *Proc. ECC'93, Groningen*, pages 1510–1513, 1993.

I. S. Krasil'shchik, V. V. Lychagin, and A. M. Vinogradov. *Geometry of Jet Spaces and Nonlinear Partial Differential Equations*. Gordon and Breach, New York, 1986.

J. Lévine and D.V. Nguyen. Flat output characterization for linear systems using polynomial matrices. *Systems & Control Letters*, 48:69–75, 2003.

Ph. Martin. *Contribution à l'Étude des Systèmes Diffèrentiellement Plats*. PhD thesis, École des Mines de Paris, 1992.

Ph. Martin, R.M. Murray, and P. Rouchon. Flat systems. In G. Bastin and M. Gevers, editors, *Plenary Lectures and Minicourses, Proc. ECC 97, Brussels*, pages 211–264, 1997.

Ph. Martin and P. Rouchon. Systems without drift and flatness. In *Proc. MTNS 93, Regensburg, Germany*, August 1993.

M. van Nieuwstadt, M. Rathinam, and R.M. Murray. Differential flatness and absolute equivalence of nonlinear control systems. *SIAM J. Control and Optimization*, 36(4):1225–1239, 1998.

P. S. Pereira da Silva. Flatness of nonlinear control systems : a Cartan-Kähler approach. In *Proc. Mathematical Theory of Networks and Systems (MTNS'2000), Perpignan*, pages 1–10, 2000.

P.S. Pereira da Silva and C. Corrêa Filho. Relative flatness and flatness of implicit systems. *SIAM J. Control and Optimization*, 39(6):1929–1951, 2001.

H. Poincaré. Sur l'uniformisation des fonctions analytiques. *Acta Mathematica*, 31:1–63, 1907. Also in Œuvres de Henri Poincaré, t. 4, pages 70–139, Gauthier-Villars, Paris, 1950.

J.W. Polderman and J.C.Willems. *Introduction to Mathematical System Theory: a Behavioral Approach*. Springer-Verlag, Berlin, 1997.

J.-B. Pomet. A differential geometric setting for dynamic equivalence and dynamic linearization. In B. Jakubczyk, W. Respondek, and T. Rzeżuchowski, editors, *Geometry in Nonlinear Control and Differential Inclusions*, pages 319–339. Banach Center Publications, Warsaw, 1993.

J.-B. Pomet. On dynamic feedback linearization of four-dimensional affine control systems with two inputs. *ESAIM-COCV*, 1997. http://www.emath.fr/Maths/Cocv/Articles/articleEng.html.

J.F. Pommaret. *Partial Differential Control Theory*. Kluwer academic publishers, 2001.

J.F. Pommaret and A. Quadrat. Localization and parametrization of linear multidimensional control systems. *Systems & Control Letters*, 37:247–269, 1999.

M. Rathinam and R.M. Murray. Configuration flatness of Lagrangian systems underactuated by one control. *SIAM J. Control and Optimization*, 36(1): 164–179, 1998.

P. Rouchon. Necessary condition and genericity of dynamic feedback linearization. *J. Math. Systems Estim. Control*, 4(2):257–260, 1994.

W.F. Shadwick. Absolute equivalence and dynamic feedback linearization. *Systems & Control Letters*, 15:35–39, 1990.

W.M. Sluis. A necessary condition for dynamic feedback linearization. *Systems & Control Letters*, 21: 277–283, 1993.

H.L. Trentelman. On flat systems behaviors and observable image representations. *Systems & Control Letters*, 19:43–45, 2004.

V.V. Zharinov. *Geometrical Aspect of Partial Differential Equations*. World Scientific, Singapore, 1992.